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Connection Between Dissipative and Resonant Conservative Nonlinear Oscillators

John McLaughlin[1]

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It is shown how to add dissipation to the "resonant" nonlinear oscillators studied by Ford and Lunsford in such a way that the system remains on the energy surface. In the dissipative system, the energy surface is stable in some directions and neutrally stable in other directions. The dissipative oscillators are special cases of the general type investigated by Sherman and McLaughlin. The connection between resonant conservative nonlinear oscillators and dissipative oscillators may make it easier to extend the theorem of Arnol'd to dissipative systems.

KEY WORDS: Attractor; conservative, dissipative, nonlinear oscillator; normal modes; resonant interactions; stochastic; threshold.

1. INTRODUCTION

Ford and Lunsford[1] have studied the nonlinear oscillators described by

$$H = \sum_{k=1}^{N} \tfrac{1}{2}\omega_k(P_k^2 + Q_k^2) + \gamma(V_3 + V_4 + \cdots) \tag{1}$$

for a certain "resonant" form of V_3 and ω_k and $V_n = 0, n > 3$. Hence V_n is a polynomial of order n in the P_k and Q_k. In Eq. (1), the ω_k and γ are real, positive constants. The resonant Hamiltonians are most easily given in action-angle variables.

$$H = \sum_{k=1}^{N} k\omega J_k + \gamma \sum_{(|n_k| + |n_l| + |n_m|) = 3} A_{klm} J_k^{|n_k|/2} J_l^{|n_l|/2} J_m^{|n_m|/2} \times \cos(n_k\phi_k + n_l\phi_l + n_m\phi_m) \tag{2}$$

$$Q_k = (2J_k)^{1/2} \cos \phi_k \tag{3}$$

$$P_k = -(2J_k)^{1/2} \sin \phi_k \tag{4}$$

[1] Physics Department, Clarkson College of Technology, Potsdam, New York.

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The sum in Eq. (2) is restricted to those terms having zero Poisson bracket with the first term. Arnol'd[2] has shown rigorously that nonresonant oscillators are near-integrable for sufficiently small energies. The numerical work of Ref. 1 strongly suggests that the converse is also true—i.e., that the resonant systems have widespread stochasticity for arbitrarily small energies.

Sherman and McLaughlin[3] have studied the model of nonlinearly coupled normal modes defined as follows:

$$dA_n/dt = -i\omega_n A_n + i \sum \gamma_{mn} A_m A_{n-m} \tag{5}$$

$$A_{-n} = A_n^* \tag{6}$$

The factor of i is inserted before the summation sign in Eq. (5) for convenience. The A_n can be thought of as representing the spectral coefficients in the expansion of some physical field obeying a quadratically nonlinear PDE in the eigenfunctions of the linearized PDE. The parameters ω_n are the eigenfrequencies of the linearized problem, which are complex in general. If ω_n^i (i.e., the imaginary part of ω_n) is positive, then infinitesimal values of A_n will increase exponentially (assuming that the nonlinear terms are negligible).

In the calculations reported in Ref. 3, all A_n, $n > 4$, were dropped and the γ_{mn} were set equal to unity. The following dispersion relation was chosen for the real part of the ω_n:

$$\omega_n^r = 0.6n + 0.12n^2 \tag{7}$$

It was found that, when ω_1^i and ω_0^i are negative and ω_2^i, ω_3^i, and ω_4^i are sufficiently positive, stochastic solutions exist. The above choice of dispersion relation for ω_n^r was made in order to produce power spectra similar to those seen in the Couette flow experiments of Swinney *et al.*[4] In unreported calculations, the authors of Ref. 3 found that in the linear dispersion case, $\omega_n^r = n\omega_1^r$, the system was strictly periodic for the range of ω_n^i studied (between zero and unity). Since the "resonant" system of Ref. 1 is also nondispersive, this result is somewhat puzzling at first. However, the model studied in Ref. 1 was conservative (ω_n real). Thus, dissipation appears to take the system off resonance in this case. There is some uncertainty in this conclusion, since the nonlinearity studied in Ref. 3 was not of Hamiltonian form.

2. CONNECTING TRANSFORMATION

In order to facilitate comparison, let us start with the specific Hamiltonian studied by Ford and Lunsford:

$$\begin{aligned} H = J_1 + 2J_2 + 3J_3 + \gamma[\alpha J_1 J_2^{1/2} \cos(2\phi_1 - \phi_2) \\ + \beta(J_1 J_2 J_3)^{1/2} \cos(\phi_1 + \phi_2 - \phi_3)] \end{aligned} \tag{8}$$

The equation of motion for J_n and ϕ_n can be obtained by writing out Hamilton's equations. The system in Eq. (8) is equivalent to the following system of coupled modes:

$$dA_1/dt = -i\omega_1 A_1 + i\Gamma(2\alpha A_1{}^* A_2 + \beta A_2{}^* A_3) \tag{9}$$

$$dA_2/dt = -i\omega_2 A_2 + i\Gamma(\alpha A_1{}^2 + \beta A_3 A_1{}^*) \tag{10}$$

$$dA_3/dt = -i\omega_3 A_3 + i\Gamma\beta A_1 A_2 \tag{11}$$

$$\omega_1 = 1, \qquad \omega_2 = 2, \qquad \omega_3 = 3 \tag{12}$$

$$A_n = \sqrt{J_n}\, e^{-i\phi_n} \tag{13}$$

$$\Gamma = -\tfrac{1}{2}\gamma \tag{14}$$

If A_4 and A_0 are dropped from the system studied in Ref. 3, α and β are set equal to unity, and ω_1, ω_2, and ω_3 are allowed to be complex, Eqs. (9)–(11) are equivalent to the system of Ref. 3 except for the factor of two in the first nonlinear term of Eq. (9). Thus, the nonlinearity of the model in Ref. 3 is not of Hamiltonian form. Of course, the model of Ref. 3 should really be compared to a four-dimensional resonant oscillator:

$$H = H_0 + 4J_4 + i\Gamma[\epsilon J_2\sqrt{J_4}\cos(\phi_4 - 2\phi_2) + \\ + \delta(J_1 J_3 J_4)^{1/2}\cos(\phi_4 - \phi_1 - \phi_3)] \tag{15}$$

In Eq. (15), H_0 is the Hamiltonian given in Eq. (8). In mode variables, this system can be written as

$$dA_1/dt = -i\omega_1 A_1 + i\Gamma(2\alpha A_1{}^* A_2 + \beta A_2{}^* A_3 + \delta A_3{}^* A_4) \tag{16}$$

$$dA_2/dt = -i\omega_2 A_2 + i\Gamma(\alpha A_1{}^2 + \beta A_3 A_1{}^* + 2\epsilon A_2{}^* A_4) \tag{17}$$

$$dA_3/dt = -i\omega_3 A_3 + i\Gamma(\beta A_1 A_2 + \delta A_1{}^* A_4) \tag{18}$$

$$dA_4/dt = -i\omega_4 A_4 + i\Gamma(\epsilon A_2{}^2 + \delta A_1 A_3) \tag{19}$$

Equations (16)–(19) differ from the system integrated in Ref. 3 even if $\alpha = \beta = \delta = \epsilon = 1$, because of the factors of two in the nonlinear terms in Eqs. (16) and (17). The mode A_0 has been dropped for brevity.

If the ω_n are chosen to be real and the terms involving A_0 are dropped, the system studied in Ref. 3 is conservative. Even though the system is not of Hamiltonian form, it seems likely that it would have stochastic solutions for arbitrarily small energies because vanishing denominators[(2)] still occur in the lowest order terms in a perturbation expansion in the nonlinearity. This point will be treated in a future publication.

3. A NEW MODEL

It is possible to modify the resonant system in such a way that, when dissipation is added to the system (i.e., when the ω_n are allowed to become complex), the system point stays on the energy surface. Furthermore, the energy surface is at least neutrally stable. In three dimensions, the generalization is as follows:

$$dA_1/dt = -i\omega_1 A_1 + i\Gamma(2\alpha A_1{}^* A_2 + \beta A_2{}^* A_3) - \kappa X A_1 \tag{20}$$

$$dA_2/dt = -i\omega_2 A_2 + i\Gamma(\alpha A_1{}^2 + \beta A_3 A_1{}^*) - \kappa X A_2 \tag{21}$$

$$dA_3/dt = -i\omega_3 A_3 + i\Gamma\beta A_1 A_2 - \kappa X A_3 \tag{22}$$

$$X \equiv |A_1|^2 + 2|A_2|^2 + 3|A_3|^2 \tag{23}$$

$$\omega_n = n + i\xi \tag{24}$$

$$\xi, \kappa \geqslant 0 \tag{25}$$

The generalization to higher dimensions is obvious. Equations (20)–(24) can be combined to yield the following two relations:

$$dx/dt = 2(\xi x - \kappa x^2) \tag{26}$$

$$dy/dt = (\xi - \kappa x)y \tag{27}$$

$$y \equiv \alpha A_1{}^2 A_2{}^* + \text{c.c.} + \beta A_1 A_2 A_3{}^* + \text{c.c.} \tag{28}$$

In Eq. (28), the symbol c.c. means complex conjugate. Taking ξ and κ to be positive, Eq. (26) has an attracting fixed point X_{fp}:

$$X_{\text{fp}} = \xi/\kappa \tag{29}$$

Thus, for appropriate choices of ξ and κ, the set of points corresponding to a given value of $X = J_1 + 2J_2 + 3J_3$ remains intact and attracts nearby points. In this sense, the energy surface becomes an attractor in the dissipative system. Furthermore, when $X = X_{\text{fp}}$, Eq. (27) shows that the quantity y is conserved. In action-angle variables, it is seen that x and y correspond to the two conserved parts of the resonant Hamiltonian:

$$x = J_1 + 2J_2 + 3J_3 \tag{30}$$

$$y = 2[\alpha J_1 \sqrt{J_2}\cos(2\phi_1 - \phi_2) + \beta(J_1 J_2 J_3)^{1/2}\cos(\phi_1 + \phi_2 - \phi_3)] \tag{31}$$

Thus, for an appropriate choice of ξ/κ, the two conserved energies are undisturbed by the addition of dissipation. The system point remains on the same four-dimensional "surface." The surface is attracting for small displacements in the x direction and neutrally stable with respect to displacements in the y direction.

The system defined by Eqs. (20)–(25) can be derived from a quadratic system of the form given in Eqs. (5) and (6) by retaining five modes, assuming that A_4 and A_0 are heavily damped ($-\omega_4{}^i \gg 1$ and $-\omega_0{}^i \gg 1$), and making suitable choices for the γ_{mn}.

4. CONCLUSION

A connection has been established between the conservative nonlinear oscillators studied by Ford and Lunsford[1] and the dissipative nonlinear coupled mode equations investigated by Sherman and McLaughlin.[3] A model has been constructed in which one may go continuously from a conservative system to a dissipative system in such a way that the constants of the motion are preserved. On the energy surface, the dissipative terms cancel out identically, so that the motion reduces to that studied by Ford and Lunsford.[1] Thus, the only effect of the dissipation is to make the energy surface an attractor. On the other hand, if the coefficients of the dissipative terms differ slightly from the values determined in Eqs. (20)–(25), the two energies are no longer conserved. It seems likely that a finite threshold for stochasticity will be produced in this case. Chirikov[5] has recently found similar behavior in his work on slightly dissipative mappings.

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